

DTP/94/116
December 1994

Coulomb effects in W^+W^- production

V.S. Fadin[†], V.A. Khoze and A.D. Martin

Department of Physics, University of Durham, Durham, DH1 3LE, U.K.

and

A. Chapovsky

Novosibirsk State University, 630090 Novosibirsk, Russia.

Abstract

We calculate the Coulomb effects on the cross section for $e^+e^- \rightarrow W^+W^-$ taking into account the instability of the W bosons. We carefully explain the consequences of instability throughout the energy range which will be accessible at LEP2. We present a formula which allows these effects to be easily implemented.

[†] Permanent address: Budker Institute for Nuclear Physics and Novosibirsk State University, 630090 Novosibirsk, Russia.

1. Introduction

The detailed measurements of the observables associated with the Z boson at LEP and SLC, together with the known values of the QED and Fermi couplings (α , G_F), have allowed precision tests of the electroweak sector of the Standard Model. If this information were supplemented by an accurate measurement of the mass of the W boson, M_W , then extremely stringent tests of the model would be possible. Much progress has been made in the measurement of the mass M_W from the study of $W \rightarrow \ell\nu$ events at the Fermilab $p\bar{p}$ collider [1], although the precision is considerably less than that for α , G_F and M_Z . One of the main objectives of the LEP2 physics programme is to improve our knowledge of M_W .

The methods which have been proposed to measure M_W from observations of the process $e^+e^- \rightarrow W^+W^-$ at LEP2 include (i) the direct reconstruction of M_W from the decay products of the W bosons, (ii) the study of the end-point of the lepton spectrum in $W \rightarrow \ell\nu$ decays and (iii) an energy scan of the cross section in the W^+W^- threshold region. Methods (i) and (iii) are expected to be most precise, although both have advantages and disadvantages. For method (i) LEP2 can be run at its highest proposed energy to maximize the event rate, but to reconstruct the $W^+W^- \rightarrow q_1\bar{q}_2q_3\bar{q}_4$ decay channels we encounter the problem of attributing all the observed decay products to the correct parent W , and thus we need to control the QCD interference (interconnection) effects [2]. These problems are reduced for the $W^+W^- \rightarrow q_1\bar{q}_2\ell\nu$ decay channels, but then the event rate is smaller and moreover an unobservable neutrino is present. The threshold scan, method (iii), is theoretically cleaner, but in the important region, within about a W width of the W^+W^- threshold, the event rate is considerably lower than that at the proposed maximum energy of LEP2. Nevertheless it has been advocated that such a scan should be done and, indeed, that it could offer the most precise determination of M_W .

The precision determination of M_W (and also the W boson width Γ_W) by means of a threshold scan relies on an accurate theoretical knowledge of the electroweak radiative corrections to the $e^+e^- \rightarrow W^+W^-$ cross section. Among these radiative effects, special attention must be paid to the electromagnetic Coulomb interaction between the W^\pm bosons, which results in the largest loop corrections in the important threshold region. It has been known for a long time [3] that when oppositely charged particles have low relative

velocity $v \ll 1$ (in units of c) Coulomb effects enhance the cross section by a factor, which, to leading order in α/v , is $(1 + \alpha\pi/v)$, provided that the particles are stable. Subsequently it was shown [4, 5] that the Coulomb effects may be radically modified when the interacting particles are short-lived rather than stable. A general prescription which shows how to account for instability effects in the threshold production of heavy particles was presented in Ref. [6]. For the particular case of instability effects in $e^+e^- \rightarrow W^+W^-$ near threshold a prescription was explicitly given in a non-relativistic framework in Ref. [7]. The results were only formulated in the non-relativistic regime and so it is desirable to present a formulation which covers the whole energy region. Such an attempt has been made in Ref. [8], but unfortunately this extension was not correct, as we shall explain in Section 4. Here we present a complete treatment of the effects of W instability on the cross section for $e^+e^- \rightarrow W^+W^-$.

In Section 2 we present a qualitative discussion of Coulomb effects in $e^+e^- \rightarrow W^+W^-$. In fact, general arguments allow us to identify the energy domain in which W instability will radically dilute the Coulomb enhancement. In Section 3 we explicitly calculate the Coulomb effects for $e^+e^- \rightarrow W^+W^- \rightarrow f_1\bar{f}_2f_3\bar{f}_4$. We first give a full non-relativistic derivation appropriate to the threshold region and then we extend the formulation to include instability effects at any energy. We discuss the level of accuracy to which these effects are included. At each stage we check that the $\Gamma_W \rightarrow 0$ limit of our results reproduces the Coulomb enhancement factor for stable W production. Whenever possible we give physical insight into the effects of W instability on W^+W^- production. Although it is more physical and transparent to use non-relativistic perturbation theory to calculate the Coulomb effects between the W^\pm bosons, in Section 4 we present an alternative derivation based on Feynman diagram techniques. Again we explain how to allow for the effects of instability of W bosons at all e^+e^- energies. In Section 5 we show how to calculate the higher-order Coulomb effects in the production of heavy unstable particles [5, 9, 10], and finally in Section 6 we present our conclusions.

2. Overview of Coulomb effects in W^+W^- production

In this paper we wish to study the effects of the Coulomb interaction between the W^+ and W^- bosons in the process

$$e^+e^- \rightarrow W^+W^- \rightarrow (f_1\bar{f}_2)(f_3\bar{f}_4) \quad (1)$$

If the W bosons were stable the effect of the Coulomb interaction on the cross section has been known for a long time [3]. We summarize this result in the subsection below, which allows us to establish notation. Then, in the following subsection we discuss the modifications which arise from the instability of the W bosons. Using general arguments we show that the modifications are particularly significant near the W^+W^- production threshold, but become negligible for c.m. energies which satisfy $\sqrt{s} - 2M_W \gg \Gamma_W$.

We shall not discuss the effects of initial state radiation. These are very important, because of the logarithmic enhancements, but they can be easily incorporated using standard structure function techniques [11] which are by now quite routine (see, for example, ref. [7]). They do not influence the qualitative features of the phenomena discussed here. The results presented below can equally well be directly applied to the process $\gamma\gamma \rightarrow W^+W^-$.

(a) $e^+e^- \rightarrow W^+W^-$: assuming stable W^\pm bosons

For future reference we first study the hypothetical case in which the W^\pm bosons are assumed to be stable, that is we switch off the interaction responsible for their decays. In this $\Gamma_W \rightarrow 0$ limit the inclusive $e^+e^- \rightarrow W^+W^-$ cross section can be written symbolically in the form

$$\sigma(s) = \sigma_0(s, M_W^2, M_W^2)(1 + \delta(R, C)) \quad (2)$$

where σ_0 is the cross section at c.m. energy \sqrt{s} in the Born approximation and $\delta(R, C)$ represents the radiative corrections. The suggestive form of the notation $\delta(R, C)$ implies that the Coulomb corrections C can be separated from the remaining radiative corrections R . In fact the separation can only be done uniquely near threshold where the two W 's are slowly moving in their c.m. frame [6, 7]. However it is just in the threshold region where the off-shell and finite width effects are most important [4, 5, 12].

For W^+W^- S-wave production the separation of the exact (all-order) Coulomb contribution and the first-order hard correction may be written in the form¹

$$1 + \delta(R, C) = |\psi(0)|^2 \left(1 + \frac{\alpha}{\pi} \delta_H(s) \right), \quad (3)$$

with the Coulomb enhancement factor $|\psi(0)|^2$ separated from the “hard” or short-distance one-loop electroweak corrections, which are denoted by $\alpha\delta_H/\pi$. The Coulomb factor, which was originally obtained by Sommerfeld and Sakharov [3], is given by

$$|\psi(0)|^2 = \frac{X}{1 - e^{-X}} = 1 + \frac{X}{2} + \dots \quad (4)$$

with $X = 2\alpha\pi/v_0$, where $\psi(0)$ is the wave function, describing the relative motion of the two W bosons, evaluated at the origin. The S-wave configuration arises from the ν exchange diagram for $e^+e^- \rightarrow W^+W^-$. Here

$$v_0 = \frac{4p_0}{\sqrt{s}} = 2\sqrt{1 - \frac{4M_W^2}{s}} \quad (5)$$

is the relative velocity of the two W bosons and p_0 is the magnitude of the 3-momentum of each W boson. The subscript 0 is to indicate that the bosons are on-mass-shell. Later we will need to introduce the off-shell velocity v and momentum p . Instead of working in terms of the c.m. energy \sqrt{s} it is more convenient to introduce an energy

$$E = \frac{p_0^2}{M_W} = \frac{s - 4M_W^2}{4M_W} \quad (6)$$

which coincides with the kinetic energy of the on-shell W bosons in the non-relativistic regime.

From (3) and (4) we see that Coulomb effects enhance the cross section by a factor which, to leading-order in α/v , is

$$\left(1 + \frac{\alpha}{v} \delta_C \right) \quad (7)$$

¹To the best of our knowledge, this type of factorized form was first proposed in Ref. [13]; see Refs. [5, 14] for subsequent discussions.

with $\delta_C = \pi$ and $v = v_0$ for *stable* W bosons. Therefore the short-distance correction in (3) is

$$\frac{\alpha}{\pi} \delta_H = \frac{\alpha}{\pi} \delta_1(s) - \frac{\alpha\pi}{v_0}, \quad (8)$$

where $\alpha\delta_1/\pi$ is the *full* one-loop electroweak correction to $e^+e^- \rightarrow W^+W^-$.

It is interesting to note from (4) that at threshold the exact Coulomb enhancement is twice the leading-order contribution. This old result is frequently overlooked in recent publications.

(b) Coulomb effects in $e^+e^- \rightarrow W^+W^- \rightarrow 4f$: qualitative discussion

The Coulomb correction is radically modified in the realistic case of unstable W bosons due to finite width and off-shell effects [7, 9]. The generic form of the cross section for $e^+e^- \rightarrow W^+W^- \rightarrow 4f$ is

$$\begin{aligned} \sigma(s) = \int_0^s ds_1 \rho(s_1) \int_0^{(\sqrt{s}-\sqrt{s_1})^2} ds_2 \rho(s_2) \sigma_0(s, s_1, s_2) (1 + \delta(R, \bar{C})) + \\ + \sum_n O\left(\alpha^n \frac{\Gamma_W}{M_W}\right) \end{aligned} \quad (9)$$

where $\sqrt{s_1}$ and $\sqrt{s_2}$ are the c.m. energies of the decay products of the W bosons, and the Breit-Wigner factors

$$\rho(s_i) = \frac{B(W \rightarrow f\bar{f})}{\pi} \frac{\sqrt{s_i} \Gamma_W(s_i)}{(s_i - M_W^2)^2 + s_i \Gamma_W^2(s_i)}. \quad (10)$$

The “running” physical width $\Gamma_W(s_i) = \sqrt{s_i} \Gamma_W / M_W$ incorporates the radiative effects associated with the decays of the W bosons. In the limit $\Gamma_W \rightarrow 0$ and $B(W \rightarrow f\bar{f}) = 1$ we see (9) reduces to (2), providing, of course, that the modified Coulomb effects, when averaged over the dominant regions of the s_1, s_2 integrations, reduce to the “stable W boson” result presented above.

The $n = 0$ term in the sum in (9) corresponds to the non-radiative background contributions to $e^+e^- \rightarrow 4f$ (and their interferences with the $e^+e^- \rightarrow W^+W^- \rightarrow 4f$ diagrams). The $n \geq 1$ terms include the radiative interferences between the W boson production and decay stages induced by n emitted quanta. Note that the separation of the terms of $O(\Gamma_W/M_W)$ is, in principle, gauge dependent. As far as we are aware the complete analytical calculation of the $O(\Gamma_W/M_W)$ interference terms has not been performed.

Apart from the appearance of the $O(\Gamma_W/M_W)$ terms, there are two modifications of the “stable W ” formula (2), in going to the realistic expression (9) in which we allow the W bosons to decay. First the obvious kinematic effect involving integrations over the Breit-Wigner forms $\rho(s_i)$, and secondly the modification (symbolically denoted by $C \rightarrow \bar{C}$) of the Coulomb interaction between the W bosons, which is our main concern.

It is straightforward to see at which e^+e^- energies, E , the modification of the Coulomb correction will be important. The typical interaction time between the W bosons is $\tau_i \sim 1/(M_W v_0^2)$, whereas the lifetime of the W bosons is $\tau \sim 1/\Gamma_W$. Therefore in the region $E \gg \Gamma_W$ we expect the Coulomb effect to be unchanged by the instability of the W bosons. To be precise, the modification could, at most, lead to a change of the cross section of $O(\Gamma_W/(M_W v_0^2))$ in this region.

On the other hand in the threshold region $E \lesssim \Gamma_W$ the Coulomb interaction time is comparable to, or even greater than, the W lifetime. We therefore anticipate that the Coulomb correction will be considerably suppressed by the instability of the W bosons. In fact we can estimate the size of the suppression of the original first-order Coulomb enhancement, $(1 + \alpha\pi/v_0)$, as follows. We note from the expression for the cross section, (9), that the interplay between the Breit-Wigner forms $\rho(s_i)$ and the phase space factor in the Born cross section σ_0 (which is proportional to the c.m. momentum p of a virtual W) suppresses contributions from the small momentum region. Indeed even in the threshold region we find

$$\langle p \rangle \gtrsim \sqrt{M_W \Gamma_W}. \quad (11)$$

Thus, contrary to the stable W case, we may say off-shell and finite width effects mask the Coulomb singularity. From (11) we see that the expansion parameter of the Coulomb series is, at most,

$$\alpha\pi\sqrt{\frac{M_W}{\Gamma_W}} \sim 0.15, \quad (12)$$

rather than $\alpha\pi/v_0$ of the stable W case. Therefore higher-order Coulomb corrections will be numerically small, although they can be calculated exactly if necessary using the Green’s function formalism of Refs. [5, 9, 10], see Section 5.

The first-order contribution to the radiative corrections $\delta(R, \bar{C})$ occurring in (9) can be written in the form

$$\delta^{(1)}(R, \bar{C}) = \frac{\alpha}{\pi} \delta_H(s) + \frac{\alpha}{v} \delta_C, \quad (13)$$

which may be compared with $\delta(R, C)$ of eq. (3) for stable W bosons. Here, in the case of unstable W bosons,

$$v = \frac{4p}{\sqrt{s}} = \frac{2}{s} \left[(s - s_1 - s_2)^2 - 4s_1s_2 \right]^{\frac{1}{2}} \quad (14)$$

and, similarly, δ_C depends on s_1, s_2 , as well as s . As before, p is the c.m. momentum of a virtual W boson and v is the relative velocity of the two bosons. The “hard” correction $\delta_H(s)$ was defined for stable W bosons in (8) for $s \geq 4M_W^2$. However in the unstable case we can have values of s below threshold. As was discussed in Ref. [7] we can safely assume the threshold value of δ_H in this region.

According to the above discussion, we do not expect the Coulomb correction $\alpha\pi/v_0$ to be modified when $E \gg \Gamma_W$. Indeed in Section 3(c) we will show that, after averaging over the dominant regions of the s_1, s_2 integrations in (9), $\langle \delta_C \rangle \approx \pi$. It is easy to see from (5) and (14) that

$$v = v_0 \left(1 + O\left(\frac{\Gamma_W}{E}\right) \right) \quad (15)$$

for $E \gg \Gamma_W$, once we note that the dominant s_1, s_2 integration regions are $|\sqrt{s_i} - M_W| \lesssim \Gamma_W$. As a result the Coulomb correction $\alpha\pi/v_0$ for stable WW production is changed at most, as a result of instability, only by effects of relative order $\Gamma_W/M_W v_0^2$ in the energy regime $E \gg \Gamma_W$.

3. Quantitative study of Coulomb effects

We start by calculating the Coulomb correction in the threshold region of W^+W^- production since this is where the problem is well-defined and, moreover, where the effects of instability of the W bosons are most important. We discuss the relativistic region at the end of the section. For unstable particles we cannot restrict ourselves to on-shell formula, and so we need to

consider Green's functions rather than matrix elements.

(a) Coulomb effects in the non-relativistic region

The Green's function with two W boson external legs is dependent on the two off-shell variables, s_1, s_2 , as well as on s . However in the non-relativistic case we can reduce the problem to the evaluation of a one-particle Green's function in an external field which depends only on one off-mass-shell variable, $p^2 \neq M_W E$. As a consequence the Coulomb factor in the matrix element for stable bosons, $\psi(0)^*$, which depends just on E , is replaced by $f(\mathbf{p}, E)$ defined by [5]

$$f(\mathbf{p}, E) = \langle \mathbf{p} | (\widehat{H} - E - i\Gamma_W)^{-1} | \mathbf{r} = 0 \rangle \left(\frac{p^2}{M_W} - E - i\Gamma_W \right) \quad (16)$$

where $|\mathbf{p}\rangle$ is a WW state of definite momentum $\mathbf{p} = \mathbf{p}(W^+) = -\mathbf{p}(W^-)$, and $|\mathbf{r}\rangle$ is a state of definite relative position. The first factor on the right-hand side of (16) is the Fourier transform of the non-relativistic Green's function $G_{E+i\Gamma_W}(\mathbf{r}', \mathbf{r})$ which describes the propagation of a W^+W^- pair created at relative distance $\mathbf{r} = 0$. The second factor ensures that $f = 1$ in the absence of the Coulomb interaction. The Hamiltonian in (16) $\widehat{H} = \widehat{H}_0 + \widehat{V}$ where

$$\widehat{H}_0 = \frac{\widehat{\mathbf{p}}^2}{M_W} \quad \text{and} \quad \widehat{V} = -\frac{\hat{\alpha}}{r}. \quad (17)$$

Before we proceed to evaluate (16), it is informative to check that it reduces to the stable, on-mass-shell result. To do this we use the Lipmann-Schwinger equations for incoming ($|\mathbf{p}_+\rangle$) and outgoing ($|\mathbf{p}_-\rangle$) states

$$|\mathbf{p}_\pm\rangle = |\mathbf{p}\rangle + \left[\left(\frac{\mathbf{p}^2}{M_W} - \widehat{H}_0 \pm i\delta \right)^{-1} \widehat{V} |\mathbf{p}_\pm\rangle \right]_{\delta \rightarrow 0}. \quad (18)$$

If we now express $|\mathbf{p}\rangle$ in terms of $|\mathbf{p}_-\rangle$ then we can readily show from (16) that the value of f for on-shell, stable W bosons is

$$\lim_{\Gamma_W \rightarrow 0} f(\mathbf{p}, E = \mathbf{p}^2/M_W) = \langle \mathbf{p}_- | \mathbf{r} = 0 \rangle = \psi_{\mathbf{p}}^-(0)^*. \quad (19)$$

Thus, since $|\psi(0)|^2 \equiv |\psi_{\mathbf{p}}^{\pm}(0)|^2$, we recover the Coulomb enhancement factor for stable W bosons².

To calculate the Coulomb modification to the cross section, as defined in (7), we expand $f(\mathbf{p}, E)$ in terms of $V(r)$. From (16) we obtain

$$\begin{aligned} 1 + \frac{\alpha M_W}{2p} \delta_C &\approx |f(\mathbf{p}, E)|^2 \\ &= 1 - 2\text{Re} \int d^3r e^{-i\mathbf{p}\cdot\mathbf{r}} V(r) G_{E+i\Gamma_W}^{(0)}(\mathbf{r}, 0) + O(V^2) \end{aligned} \quad (20)$$

where $G^{(0)}$ is the free-particle Green's function

$$\begin{aligned} G_{E+i\Gamma_W}^{(0)}(\mathbf{r}, 0) &= \langle \mathbf{r} | \left(\widehat{H}_0 - E - i\Gamma_W \right)^{-1} | 0 \rangle \\ &= \frac{M_W}{4\pi r} \exp(-\kappa r) \end{aligned} \quad (21)$$

with

$$\kappa = \sqrt{M_W(-E - i\Gamma_W)} \equiv p_1 - ip_2. \quad (22)$$

Solving for the real and imaginary parts, we have

$$p_{1,2} = \left[\frac{1}{2} M_W \left(\sqrt{E^2 + \Gamma_W^2} \mp E \right) \right]^{\frac{1}{2}}, \quad (23)$$

with E given by (6). We insert the Green's function (21) into (20), and perform the angular integration. We obtain

$$\alpha \delta_C = -2 \int_0^\infty dr V(r) e^{-p_1 r} \{ \sin(p + p_2)r + \sin(p - p_2)r \} \quad (24)$$

where, at this stage, we have left $V(r)$ arbitrary so that we will be better able to draw attention to the specific properties of the Coulomb potential $V(r) = -\alpha/r$.

(b) Physical interpretation

Before we present the analytical result of the integration in (24), it is informative to discuss some interesting features of δ_C . Indeed the interpretation of result (24) for the Coulomb correction δ_C is subtle and needs careful

²The Coulomb interaction between incoming particles enhances the matrix element by a factor $\psi_{\mathbf{p}}^+(0)$, while an interaction between outgoing particles gives a factor $\psi_{\mathbf{p}}^-(0)^*$.

explanation. We begin by assuming that the W bosons are stable, $\Gamma_W = 0$ and $E > 0$. Then from (23) we have

$$p_1 = 0, \quad p_2 = p_0 \equiv \sqrt{M_W E}. \quad (25)$$

In this case (24) can be readily evaluated using

$$\frac{2}{\pi} \int_0^\infty \frac{dx}{x} \sin ax = \text{sgn } a \equiv \begin{cases} 1 & \text{for } a > 0 \\ 0 & \text{for } a = 0 \\ -1 & \text{for } a < 0, \end{cases} \quad (26)$$

which yields

$$\delta_C = \pi \{1 + \text{sgn}(p - p_0)\}. \quad (27)$$

We see that δ_C is a non-analytic function of the virtuality $(p - p_0)$, with a *discontinuity* at the mass shell value $p = p_0$.

The non-analytic behaviour of δ_C is a consequence of the long-range nature of the Coulomb force, as we can verify by truncating the potential so that $V(r) = 0$ for $r > R_0$, with $p_0 R_0 \gg 1$. For the truncated potential, we find that δ_C makes a *smooth* transition from 0 to 2π and that most of the variation occurs while the virtuality $(p - p_0)$ covers the range $\sim -1/R_0$ to $\sim 1/R_0$. Note that the dominant contribution to the integral (26) comes from values $x \lesssim 1/|a|$.

We conclude that *any* non-zero virtuality will drastically change the on-shell value $\delta_C = \pi$. This is contrary to explicit claims presented in Ref. [8].

Now let us study the effect of the finite width Γ_W of the W bosons. From (24) we see that the width plays the role of a cut-off on the potential at distances $R_0 \sim 1/p_1$, where $p_1 \sim \Gamma_W \sqrt{M_W/E}$ or $\sqrt{M_W \Gamma_W}$ according to whether E is greater or less than Γ_W . Thus, from the above discussion, we anticipate that the presence of the finite width will lead to a smooth transition between $\delta_C = 0$ and $\delta_C = 2\pi$ which occurs dominantly in the region

$$|p - p_2| \sim p_1. \quad (28)$$

The non-zero W width thus restores the analyticity of δ_C as a function of the W boson virtuality.

Armed with this understanding, we return to (24) and carry out the integration explicitly. We find

$$\delta_C = \pi - 2 \arctan \left(\frac{p_1^2 + p_2^2 - p^2}{2pp_1} \right), \quad (29)$$

a result which was obtained in Ref. [7] by a different approach. Formula (29) embodies all the special features of δ_C that we have discussed above.

(c) Coulomb effects for $E \gg \Gamma_W$

In Section 2(b) we used physical arguments to show that the instability of W bosons would not change the Coulomb enhancement factor of the $e^+e^- \rightarrow W^+W^-$ cross section for energies $E \gg \Gamma_W$. At first sight formula (29) seems to contradict this claim, because δ_C does not equal π for $E \gg \Gamma_W$. However to determine the possible change of *cross section* we must integrate the Coulomb correction δ_C over the W boson virtualities s_1, s_2 as in (9)³. In the non-relativistic case this reduces to an integration over the single off-shell variable $p^2 \approx (\sqrt{s} - \sqrt{s_1} - \sqrt{s_2})M_W$ [5]

$$\int_0^s ds_1 \rho(s_1) \int_0^{(\sqrt{s}-\sqrt{s_1})^2} ds_2 \rho(s_2) \approx \int_0^\infty dp^2 \frac{\Gamma_W}{\pi M_W [(p^2/M_W - E)^2 + \Gamma_W^2]}. \quad (30)$$

For $E \gg \Gamma_W$ the arctan modification of δ_C of (29) is an *odd* function of the virtuality $(p^2 - M_W E)$ in the dominant region of the p^2 integration, which is specified by $|p^2 - M_W E| \lesssim M_W \Gamma_W$, and hence integrates to zero. To be explicit, for $E \gg \Gamma_W$ we find

$$\delta_C \approx \pi - 2 \arctan \left(\frac{M_W E - p^2}{M_W \Gamma_W} \right) \quad (31)$$

in the essential p^2 region, and the difference of δ_C from the on-mass-shell value of π averages to zero when integrated over the Breit-Wigner form in (30).

³The off-shell behaviour of the Coulomb effects in the unintegrated cross section would be interesting to observe, but, in practice, there will be insufficient statistics for such a study.

(d) Closed cross-section formula in the non-relativistic limit

In the non-relativistic limit it is possible to use a mathematical trick to carry out the off-shell integration and so obtain a closed formula for the Coulomb corrections to the cross section. We wish to perform the off-shell integration of (30) over $|f(\mathbf{p}, E)|^2$ weighted by the phase space factor p/M_W occurring in σ_0 . We notice that the normalization factor for $f(\mathbf{p}, E)$ given in (16) cancels the Breit-Wigner denominator so that

$$\begin{aligned}
I_C &\equiv \int_0^\infty \frac{p dp^2 \Gamma_W |f(\mathbf{p}, E)|^2}{\pi M_W^2 [(p^2/M_W - E)^2 + \Gamma_W^2]} \\
&= \frac{4\pi}{M_W^2} \int \frac{d^3 p}{(2\pi)^3} \Gamma_W \left| \langle \mathbf{p} | (\widehat{H} - E - i\Gamma_W)^{-1} | \mathbf{r} = 0 \rangle \right|^2 \\
&= \frac{4\pi}{M_W^2} \langle \mathbf{r} = 0 | \frac{1}{\widehat{H} - E + i\Gamma_W} \Gamma_W \frac{1}{\widehat{H} - E - i\Gamma_W} | \mathbf{r} = 0 \rangle \\
&= \frac{4\pi}{M_W^2} \text{Im } G_{E+i\Gamma_W}(0, 0) \\
&= \frac{p_2}{M_W} + \alpha \arctan\left(\frac{p_2}{p_1}\right) + O(\alpha^2), \tag{32}
\end{aligned}$$

where $p_{1,2}$ are defined in (22). Here we have used the completeness relation over the $|\mathbf{p}\rangle$ states and the explicit expression for the Green's function which can be found in Ref. [4].

Formula (32) clearly demonstrates that the modification of the cross section due to W instability is small in the region $E \gg \Gamma_W$, since in this region

$$p_2 \approx p_0, \quad p_1 \approx (\Gamma_W/E)p_0. \tag{33}$$

Thus the arctan approaches $\pi/2$ and hence (32) becomes up to accuracy $O(\Gamma_W/E)$

$$I_C \approx \frac{1}{2} v_0 \left(1 + \frac{\pi\alpha}{v_0} \right) \tag{34}$$

which corresponds to the leading-order stable boson result.

(e) Instability effects in the relativistic region

The Coulomb correction is not uniquely defined and gauge independent in

the relativistic region. For the production of stable W bosons the Coulomb correction could equally well be taken as

$$\frac{\alpha\pi}{v_0} \quad \text{or} \quad \alpha\pi \left(\frac{M_W}{2p_0} \right) \quad (35)$$

for example. In the relativistic domain we should not, therefore, discuss the influence of W instability on the *Coulomb* correction but rather its possible modification of the *total* first-order correction. However once we have agreed to *define* the Coulomb expansion parameter X of (4) as $2\alpha\pi/v_0$ and the “hard” scattering correction δ_H of (3) by (8), we need only focus on the modification of the $\alpha\pi/v_0$ term. According to (8) and (13) the modification, at leading order, is defined by adding the expression

$$\frac{\alpha}{v} \delta_C - \frac{\alpha}{v_0} \pi \quad (36)$$

to the correction for stable W production. This difference gives, after the integration in (9), the modification of the cross section due to the instability of the W bosons.

We have calculated δ_C in the non-relativistic domain where the Coulomb correction is well defined and found that for $E \gg \Gamma_W$ the modification is, at most, of relative order Γ_W/E . Thus it is evident that the modification of the cross section in the relativistic domain $E \gtrsim M_W$ will be of relative order $\alpha\Gamma_W/M_W$, at most, which is beyond our accuracy.

It is easy to check that if δ_C in the relativistic domain is defined by (29) it will satisfy the above criteria, *provided* that the relativistic expressions (5) and (14) are used for v_0 and v respectively, with E defined by (6).

4. Alternative derivation of the Coulomb effects

The calculation of the Coulomb correction that was presented in Section 3 was based on non-relativistic perturbation theory. We believe this approach is the most physical and transparent. Moreover the Coulomb interaction is only well defined in the non-relativistic domain and, as we have seen, this is the region in which the effects of instability are important. Nevertheless it is informative to present an alternative derivation based on Feynman diagram techniques.

In the Feynman diagram approach we have to evaluate the integral

$$I = -i \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 + i\varepsilon)[(k - p_-)^2 - M_W^2 + iM_W\Gamma_W]} \times \frac{1}{[(k + p_+)^2 - M_W^2 + iM_W\Gamma_W]} \quad (37)$$

which corresponds to the loop diagram in which a photon of 4-momentum k is exchanged between the two outgoing W bosons of 4-momenta p_{\pm} , that is $s_1 = p_+^2$ and $s_2 = p_-^2$. We use Feynman parameter techniques to reduce this integral to the form

$$I = \frac{-1}{8\pi^2 s} \int_{-1}^1 \frac{dx}{(x + (s_1 - s_2)/s)^2 - \frac{1}{4}v^2} \times \log \left(\frac{2\kappa^2 + \frac{1}{2}sx^2}{2M_W^2 - s_1 - s_2 - 2iM_W\Gamma_W - (s_1 - s_2)x} \right) \quad (38)$$

where κ is defined in (22) and v is given by (14). For arbitrary values of v the result of the integration (38) cannot be expressed in terms of elementary functions, but instead involves a sum of Spence functions (see, e.g. [8]). However we do not require the general expression for I since it contains, besides the Coulomb effect, other contributions which involve infrared singularities etc. The extraction of the Coulomb part can only be done unambiguously in the non-relativistic limit, and then only with an accuracy up to terms of relative order v . Now it is clear from (38) that we can extend the region of integration from $(-1, 1)$ to $(-\infty, \infty)$ without changing the result in the non-relativistic limit, provided that $(s_1 - s_2)/s$ is small enough to preserve the $1/v$ Coulomb singularity. Then the integral can be readily evaluated by making use of the analytic properties of the integrand. First we note that the zeros of the denominator in (38) are cancelled by the logarithm, since its argument becomes unity at these points. The only singularities of the integrand of (38) which remain are the branch points at $x = \pm 2i\kappa/\sqrt{s}$ and at

$$x = x_0 \equiv \frac{2M_W^2 - s_1 - s_2 - 2iM_W\Gamma_W}{s_1 - s_2}. \quad (39)$$

We see that the position of x_0 depends on the sign of $s_1 - s_2$. If $s_1 - s_2$ is positive (negative) then x_0 lies in the lower (upper) half plane. This

means that the x_0 branch point moves across the path of integration and that our approximation (of extending the range of integration) has destroyed the original analyticity of I as a function of $(s_1 - s_2)$. We shall see later some interesting consequences of this observation.

We can now readily evaluate the integral I of (38). If $(s_1 - s_2) > 0$ we deform the contour of integration around the cut starting at the branch point $x = 2i\kappa/\sqrt{s}$, and for $(s_1 - s_2) < 0$ we wrap the contour around the cut starting from $x = -2i\kappa/\sqrt{s}$. The result is

$$I = \frac{1}{4\pi i v s} \log \left(\frac{i\kappa + \frac{1}{2}\sqrt{s}\Delta + p}{i\kappa + \frac{1}{2}\sqrt{s}\Delta - p} \right) \quad (40)$$

where p is given by (14) and

$$\Delta \equiv |s_1 - s_2|/s. \quad (41)$$

Since the first-order correction to the matrix element is proportional to I we are only interested in the real part of I . In fact the Coulomb correction is

$$\delta_C^F = -8\pi v s \operatorname{Re} I = \pi - 2 \arctan \left(\frac{|i\kappa + \frac{1}{2}\sqrt{s}\Delta|^2 - p^2}{2pp_1} \right) \quad (42)$$

where $\kappa \equiv p_1 - ip_2$ is given in (22). We use the superscript F to distinguish the δ_C^F obtained from Feynman diagram techniques from the δ_C which we calculated using non-relativistic perturbation theory in (29). Apart from the occurrence of Δ in (42), the two results coincide.

A result identical to (42) was obtained by Bardin, Beenakker and Denner [8], also using the Feynman diagram approach. The two formulae for δ_C^F can be seen to be the same if we note that their $\beta_M = 2i\kappa/\sqrt{s}$. However the wrong conclusions were drawn in Ref. [8], since Δ should be set to zero, as we will show below. An alert reader may have already guessed that this would be the case. If we were to retain Δ then we would have a spurious singularity along the line $s_1 = s_2$ (cf. (41)); a singularity which was introduced by the approximation used to evaluate the integral I of (38).

In summary we have presented two derivations of the Coulomb correction. One based on non-relativistic perturbation theory which gives δ_C of (29), see also Ref. [7], and another based on Feynman diagram techniques which gives δ_C^F of (42), see also Ref. [8]. The answers agree, except for the appearance of

Δ in δ_C^F . Both methods are, of course, equally correct; the difference arises because of the approximations made in the derivation.

All these results have been obtained in the approximation that $v \ll 1$. Now in the dominant range of the s_1, s_2 integrations in (9) we see that

$$\Delta \equiv \frac{|s_1 - s_2|}{s} \sim \frac{\Gamma_W}{M_W} \lesssim \langle v \rangle^2, \quad (43)$$

recall eq.(11). So, on inspection of (40) and (42) we see that Δ may be neglected in comparison with $i\kappa/\sqrt{s}$ and p/\sqrt{s} . Therefore the two approaches, yielding δ_C and δ_C^F , are entirely consistent in the non-relativistic region, as indeed they must be. Thus whether or not we choose to retain Δ might appear to be harmless. This is true near threshold, but to extend the result away from the $v_0 \ll 1$ region we must investigate the higher-order terms in v to ensure that we recover the formula for stable W bosons in the limit that $\Gamma_W \rightarrow 0$.

We rewrite the cross-section formula (9) in terms of the variables

$$x_i = \frac{s_i - M_W^2}{M_W \Gamma_W}, \quad \text{with } i = 1, 2. \quad (44)$$

If we then insert eq.(13) for $\delta(R, \bar{C})$ we obtain

$$\begin{aligned} \lim_{\Gamma_W \rightarrow 0} \sigma(s) &= \sigma_0(s) \left(1 + \frac{\alpha}{\pi} \delta_1(s) \right) + \\ &+ \sigma_0(s) \frac{\alpha}{v_0} \int_{-\infty}^{\infty} \frac{dx_1}{\pi(x_1^2 + 1)} \int_{-\infty}^{\infty} \frac{dx_2}{\pi(x_2^2 + 1)} (\delta_C(\Gamma_W = 0) - \pi) \end{aligned} \quad (45)$$

where by $\delta_C(\Gamma_W = 0)$ we mean either δ_C of (29) or δ_C^F of (42) expressed in terms of x_1, x_2 so that the integrals can be performed in the limit $\Gamma_W \rightarrow 0$. To evaluate δ_C we use (14), (23) and (6) and find

$$(\delta_C(\Gamma_W = 0) - \pi) = -2 \arctan \left(\frac{x_1 + x_2}{2} \right). \quad (46)$$

The integration of this term gives zero and we recover the stable W boson result. Recall that $\alpha\delta_1/\pi$ is the *full* one-loop correction, see (8). On the other hand for δ_C^F we find

$$(\delta_C^F(\Gamma_W = 0) - \pi) = -2 \arctan \left(\frac{x_1 + x_2}{2} + \frac{v_0}{2} \frac{|x_1 - x_2|}{2} \right) \quad (47)$$

and the $\Gamma_W \rightarrow 0$ limit no longer reproduces the stable W boson formula, as it should. In fact it is easy to see that in this case the result gives a value that is smaller than the Coulomb correction $\alpha\pi/v_0$ for stable W bosons. For $v_0 \ll 1$ it gives errors of relative order v_0 , but in the relativistic limit $\frac{1}{2}v_0 \rightarrow 1$ it gives a Coulomb correction of $\frac{2}{3}(\alpha\pi/v_0)$ instead of $\alpha\pi/v_0$. This explains one of the anomalies in Table 1 of Ref. [8].

The conclusion is that we must set $\Delta = 0$ and so formula (29), which was given previously in Ref. [7], is correct for all energies, provided that the appropriate relativistic definitions are used for the kinematic variables⁴.

5. Higher-order Coulomb corrections

At the outset we should emphasize that the higher-order Coulomb corrections to $e^+e^- \rightarrow W^+W^-$ are small and their exact (all-order) calculation is beyond the needs of LEP2 today. The total contribution is less than the existing uncertainties in the calculations of the $O(\alpha)$ “standard” electroweak effects. Nevertheless, since Coulomb physics (which is associated with large space-time intervals) is so different from the other radiative effects, it merits study in its own right.

Section 3(a) already contains the appropriate formalism for calculating the all-order Coulomb effect between unstable W bosons. In analogy to (3), the correction factor is

$$1 + \delta(R, \bar{C}) = |f(\mathbf{p}, E)|^2 \left(1 + \frac{\alpha}{\pi} \delta_H(s) \right) \quad (48)$$

where $\delta_H(s)$ is defined by (8) and the non-relativistic expression for $f(\mathbf{p}, E)$ is given in (16). Up to a factor $(p^2/M_W - E - i\Gamma_W)$, $f(\mathbf{p}, E)$ is the Fourier transform of the non-relativistic Green’s function; that is

$$f(\mathbf{p}, E) = \left(\frac{p^2}{M_W} - E - i\Gamma_W \right) \int d^3r \, e^{-i\mathbf{p}\cdot\mathbf{r}} G_{E+i\Gamma_W}(\mathbf{r}, 0) \quad (49)$$

where

$$G_{E+i\Gamma_W}(\mathbf{r}, \mathbf{r}') = \langle \mathbf{r} | \left(\widehat{H} - E - i\Gamma_W \right)^{-1} | \mathbf{r}' \rangle. \quad (50)$$

⁴Unfortunately the authors of Ref. [8] used the formula in an incorrect way, which resulted in a misleading comparison in their Table 1.

This Fourier transform can be calculated with the help of the Meixner representation [15] of the Green's function $G_E(\mathbf{r}, 0)$. It is found to be [10]

$$f(\mathbf{p}, E) = 1 + 2\alpha M_W \kappa \int_0^1 dx \frac{x^{-(\frac{1}{2}\alpha M_W/\kappa)}}{\kappa^2(1+x)^2 + p^2(1-x)^2} \quad (51)$$

where κ is defined in (22). It can be shown that the integral in (51) is convergent for all real values of E , provided that $\Gamma_W > 3\sqrt{3}M_W\alpha^2/32$ — a condition easily satisfied by the W boson. Therefore the representation (51) is applicable and well convergent for all real values of E , both below and above the WW threshold. The integrand in (51) has no singularities in the interval $0 < x < 1$ and so $f(\mathbf{p}, E)$ can be readily computed numerically. Moreover we may expand $f(\mathbf{p}, E)$ as a power series in α . It is easy to check that the real part of the leading term is $\alpha M_W \delta_C/4p$ with δ_C given by (29).

The identification of the leading term is the key to the generalisation of representation (51) to the relativistic case. It can be done by making the replacement $M_W \rightarrow \frac{1}{2}\sqrt{s}$ and by using the correct relativistic expression for p (cf. (14)), together with κ defined in (22) and E given by (6).

6. Conclusions

The process $e^+e^- \rightarrow W^+W^-$ is one of the most fundamental reactions to be studied at LEP2. It provides a unique opportunity to probe the heart of the Standard Model, particularly if a precise measurement of the mass M_W of the W boson can be obtained. For the measurement of M_W it is necessary to have an accurate theoretical knowledge of the cross section, especially in the region of the WW threshold.

Here we calculate the corrections to the cross section for $e^+e^- \rightarrow W^+W^-$ which arise from the instability of the produced W bosons. Although our result applies at all energies it is useful to concentrate on the modification which occurs in the important W^+W^- threshold region. For stable W bosons we may write the cross section in the symbolic form

$$\begin{aligned} \sigma &\sim f_{\text{ISR}} f_{\text{EW}} |\psi(0)|^2 \sigma_0 \\ &\sim v_0^\lambda \left(1 + \frac{\alpha}{\pi} \delta_H\right) \left(1 + \frac{\alpha\pi}{v_0} + \dots\right) v_0 \end{aligned} \quad (52)$$

where v_0 is the relative velocity of the W bosons and σ_0 is the Born cross section. Since initial state radiation (ISR) can be included in a straightforward way we have regarded it as an inessential complication and neglected it in our study. For completeness we show its threshold behaviour in (52), where $\lambda \approx 4\alpha \log(s/m_e^2)/\pi$. The “hard” or short-distance electroweak (EW) corrections are also noted in (52) for completeness. However the threshold behaviour is dominated by the Coulomb enhancement factor and the v_0 phase space factor in σ_0 .

Clearly in order to precisely measure M_W by an energy scan of the cross section in the threshold region it is crucial to calculate the modification of the Coulomb effect arising from the instability of the W bosons. It is easy to see that the major modification will, in fact, occur in the threshold region. Essentially what happens is that the instability of the W bosons smooths out the Coulomb singularity on account of the intrinsic uncertainty in their relative velocity $\sim \sqrt{\Gamma_W/M_W}$. We quantify the modification due to instability in eqs. (9), (13) and (29).

The modification is very important for energies $E \lesssim \Gamma_W$, but fades away as E increases so that for $\Gamma_W \ll E \ll M_W$ the effect is of $O(\Gamma_W/E)$ at most, and for $E \gtrsim M_W$ of $O(\Gamma_W/M_W)$ at most, where the energy E is defined in (6). The formula that we present for the modification of the cross section due to the instability of the W bosons is unambiguous for all energies⁵, despite the fact that the Coulomb interaction can only be uniquely defined for $E \ll M_W$. Using the formulae (9), (13) and (29) it is straightforward to allow for the important W boson instability effects in an experimental study of $e^+e^- \rightarrow W^+W^-$ in the WW threshold region.

Acknowledgements

We thank W.J. Stirling for valuable discussions. V.S. Fadin thanks the Centre for Particle Theory and Grey College of the University of Durham for

⁵Clearly we must use the relativistic expressions for the kinematic variables, that is (14) and (23) with E defined as in (6). We have shown in Section 4 that the variable Δ introduced in Ref. [8] must be set to zero. Δ is an artefact of the approximation used to calculate the Coulomb effects and introduces a spurious singularity, but, more important, it leads to an increasingly incorrect result as the energy increases away from the threshold region.

hospitality. The financial support of the UK Particle Physics and Astronomy Research Council is gratefully acknowledged.

References

- [1] CDF Collaboration: Young-Kee Kim, to be published in Proc. of 27th International Conference on High Energy Physics, eds. P.J. Bussey and I.G. Knowles, Glasgow, July 1994.
- [2] T. Sjöstrand and V.A. Khoze, *Z. Phys.* **C62** (1994) 281; *Phys. Rev. Lett.* **72** (1994) 28.
- [3] A. Sommerfeld, “Atombau und Spektrallinien”, Bd. 2 (Vieweg, Braunschweig, 1939); A.D. Sakharov, *JETP* **18** (1948) 631.
- [4] V.S. Fadin and V.A. Khoze, *JETP Lett.* **46** (1987) 525; *Sov. J. Nucl. Phys.* **48** (1988) 309; **53** (1991) 692.
- [5] V.S. Fadin and V.A. Khoze, Proc. of 24th LNPI Winter School, Leningrad, Vol. 1, p. 3 (1989).
- [6] V.S. Fadin, V.A. Khoze and A.D. Martin, *Phys. Lett.* **B320** (1994) 141.
- [7] V.S. Fadin, V.A. Khoze and A.D. Martin, *Phys. Lett.* **B311** (1993) 311.
- [8] D. Bardin, W. Beenakker and A. Denner, *Phys. Lett.* **B317** (1993) 213.
- [9] V.S. Fadin, V.A. Khoze and A.D. Martin, *Phys. Rev.* **D49** (1994) 2247.
- [10] V.S. Fadin, V.A. Khoze and M.I. Kotsky, *Z. Phys.* **C64** (1994) 45.
- [11] E.A. Kuraev and V.S. Fadin, *Sov. J. Nucl. Phys.* **41** (1985) 466.
- [12] M.J. Strassler and M.E. Peskin, *Phys. Rev.* **D43** (1991) 1500; Y. Sumino et al., *Phys. Rev.* **D47** (1992) 56; M. Jeżabek, J.H. Kühn and T. Teubner, *Z. Phys.* **C56** (1992) 653; M. Jeżabek and T. Teubner, *Z. Phys.* **C59** (1993) 669.

- [13] J. Harris and L.M. Brown, *Phys. Rev.* **105** (1957) 1656.
- [14] V.S. Fadin, V.A. Khoze and T. Sjöstrand, *Z. Phys.* **C48** (1990) 613.
- [15] J. Meixner, *Math. Z.* **36** (1933) 677.